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NOTE

A TRADEOFF THEOREM FOR SPACE AND REVERSAL

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Abstract. In this note, the following result is proved: Let M be an off-line Turing machine, n the input length, S the work space and R the total number of reversals of the work tape heads. Then $RS = o(n)$ implies $RS = O(1)$.

1. Introduction

There are many tradeoffs in the phenomenon of computation but only a few results have been obtained. Some results are based on a special model rather than a Turing machine. Some results are only concerned with a special problem such as sorting. Perhaps the most significant tradeoff result belongs to Borodin and Cook [1]. They proved that on a general unrestricted sequential model of computation, Time · Space $= \Omega(N^2/\log N)$, in order to sort N elements, each in the range $[1, N^2]$.

Rackoff and Dymond noticed another kind of tradeoff phenomenon, space versus reversal. Except for some very simple languages, if one wants to reduce the space, then one has to use more reversals, and vice versa. They conjectured that if $RS = o(n)$, then the language accepted by the Turing machine is regular. Take $L = \{0^n 1^n\}$ for example. If we count the number of 0's in unary, we have to use $O(n)$ work space. If we count in binary, we have to use $O(n)$ work tape reversals. The product of space and reversal is at least n. Rackoff even suggested that we can prove this conjecture by using a similar method of proving that $S = o(\log \log n)$ implies $S = O(1)$ [2].

The present author proved this tradeoff conjecture, i.e., that $RS = o(n)$ implies $S = O(1)$. Therefore, the language accepted by this Turing machine is regular. Furthermore, the present author proved that $RS = o(n)$ implies $R = O(1)$. Thus the result is symmetric: $RS = o(n)$ implies $RS = O(1)$. We can view it as a gap theorem as well.

2. The main result

Theorem 1. *Let M be an off-line deterministic Turing machine* [2] *having one read-only input tape and k work tapes. Let n be the input length, S the work space used and R the total number of worktape head reversals. Then $RS = o(n)$ implies $RS = O(1)$.*

Proof. A phase is a period of the computation during which no worktape head changes its move direction. Since, in every phase, the number of the worktape head movements is less than or equal to S, the total number of movements of worktape heads during the whole computation is less than or equal to $RS = o(n)$.

Let w be any fixed input of length n. We call the ith input square 'marked' if there is a moment at which the input head is scanning the ith input square and at least one worktape head is going to move left or right. Let I be an interval in which no square is marked. In other words, whenever the input head is inside I, the worktape head does not move at all.

Now consider a pair of squares (c_1, c_2) in I, where $c_1 < c_2$. Suppose that in some stage of the computation the input head is at square $c_1 - 1$, machine M is in state q and the k worktape heads are scanning symbols $\Delta_1, \ldots, \Delta_k$ respectively. Suppose also that the input head moves a square right, and the input head does not leave the area (c_1, c_2) until it comes back to $c_1 - 1$ from the right-hand side; this time M enters state q', and the worktape heads scan symbols $\Delta'_1, \ldots, \Delta'_k$ respectively (notice that these worktape heads do not move at all during this period). We use the following notation to express this situation:

$$\text{Left}, q, (\Delta_1, \ldots, \Delta_k) \to \text{Left}, q', (\Delta'_1, \ldots, \Delta'_k).$$

If the input head leaves the area (c_1, c_2) from the right-hand side, that is, it comes to $c_2 + 1$, we express this by

$$\text{Left}, q, (\Delta_1, \ldots, \Delta_k) \to \text{Right}, q', (\Delta'_1, \ldots, \Delta'_k).$$

In the same way, if input head enters this area (c_1, c_2) from the right-hand side, and leaves from left or right, we write

$$\text{Right}, q, (\Delta_1, \ldots, \Delta_k) \to \text{Left}, q', (\Delta'_1, \ldots, \Delta'_k)$$

or

$$\text{Right}, q, (\Delta_1, \ldots, \Delta_k) \to \text{Right}, q', (\Delta'_1, \ldots, \Delta'_k)$$

respectively.

All these notations constitute a characteristic table of (c_1, c_2). Notice that there are only finite, say C, different characteristic tables.

We prove that for any input word w, if there is an interval of length $C+1$ in which there is no marked squares, then we can find a shorter input w which uses the same number of work tape movements as w.

Assume that this interval is $(i, i+C+1)$. Consider the following $C+1$ different intervals: $(i, i+C+1), (i+1, i+C+1), \ldots, (i+C, i+C+1)$. There must be at least two $j_1 < j_2$, such that the following two intervals have the same characteristic table:

$$(j_1, i+C+1), \qquad (j_2, i+C+1).$$

Suppose that the input is $w = a_1a_2a_3 \ldots a_n$. Set

$$w' = a_1a_2 \ldots a_{j_1-1}a_{j_2} \ldots a_n.$$

It is easy to see that w' and w_1 have the same work tape movements. Now set

$$S_m = \{w \mid M \text{ makes exactly } m \text{ movements of the work tape heads on } w\}.$$

If for sufficiently large m, $S_m = \emptyset$, then we have $RS = O(1)$ already. Therefore, we consider the case that there are infinitely many m such that $S \neq \emptyset$. For these m define

$$L_m = \min\{|w| \mid w \in S_m\}.$$

We will show that $L_m \leq (C+1)(m+1)$.

If $L_m > (C+1)(m+1)$, then there is an interval of length $\geq C+1$ in which there is no marked square at all. As above, we can shorten it to obtain a new input which has the same work tape movements. In other words, the new input is still in S_m, a contradiction to the definition of L_m. Therefore, we must have $L_m \leq (C+1)(m+1)$. Remember that RS is not less than the total number of worktape movements. We have therefore

$$RS \geq m \geq L_m/(C+1) - 1 = |w|/(C+1) - 1 = n/(C+1) - 1$$

for infinitely many m. This contradicts the fact that $RS = o(n)$. Therefore, we have proved that for sufficiently large m, $S_m = \emptyset$ and therefore, $RS = O(1)$. □

3. Generalization

The above theorem can be generalized to some other sequential computation models (it is trivial for parallel models). But before showing this, we have to define the conception reversal for them. According to a general principle in [3], the reversal is the total number of phases in the whole computation, and a phase is a period of the computation during which no information written on the work space is read later in the same period. We consider several examples.

(1) MMMTM. The off-line multi-tape multi-head multi-dimensional Turing machine has a read-only one-dimensional one-head input tape and several multi-head multi-dimensional worktapes. One phase is a period of the computation during which no worktape cell is entered twice or more by worktape heads.

(2) SMM [4]. The storage modification machine has a finite control, a read-only tape (tape 0) and k worktapes (tape $1, 2, \ldots, k$), just like a multi-tape Turing machine. Each tape cell has two heads (L and R) connecting to its left and/or right neighbours.

According to its state and the symbols of the cells that its heads are pointing to, the machine can (1) change to a new state, (2) write down a symbol on the cell that its ith worktape head is pointing to, (3) move its (input and work) tape heads from cells to their neighbours (we use L, R, S to denote the directions of the movements), and (4) move the L, R heads of the cell that the ith ($i=1,2,\ldots,k$) head of the finite control is pointing to, to the cell that the jth ($j=1,2,\ldots,k$) head of the finite control is pointing to.

Let $K=\{1,2,\ldots,k\}$ be the set of the worktape heads; then an SMM can be represented by a mapping

$$Q\times\Sigma^{K+1}\to Q\times\Sigma^{K}\times\{L,R,S\}^{K+1}\times K^{2K}.$$

A phase is a period during which no work cell is entered twice or more by the heads of the finite control. The space is the total number of work cells used.

For these two models, the number of work tape head movements is not more than $RS=o(n)$. Our proof goes well. Hence we have the additional result.

Theorem 2. *Let M be an off-line multi-tape multi-head multi-dimensional Turing machine or a storage modification machine. Let n be the input length, S the work space used and R the total number of reversals on the work space. Then $RS=o(n)$ implies $RS=O(1)$.*

References

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